My Proofs of the Laws of Logarithms

Albert Dorador

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This document presents step-by-step proofs of the Laws of Logs by a non-mathematician, using a variant of Sir Isaac Newton's proof template, which I feel is just as elegant-looking as convenient.

Constructio
Consider any arbitrary a $\in \mathbb{R}^+ \backslash \{1\}$, and arbitrary $x, y \in \mathbb{R}^+$.
It is known that $Domain(a^x) = \mathbb{R}$, and that $Range(a^x) = \mathbb{R}^+$.
In addition, it is known that $Range\,(log_a x) = \mathbb{R}$, and $Domain(log_a x) = \mathbb{R}^+$.
Therefore, $a^{log_a x}$, as well as $a^{log_a y}$ exist.

Propositio

$$log_a x + log_a y = log_a(xy)$$

.

Demonstratio
Consider the quantity $a^{log_a x} \cdot a^{log_a y}$.
By a power law, we know that

$$a^{log_a x} \cdot a^{log_a y} = a^{log_a x + log_a y}$$

.

But, by the definition of the exponential and logarithmic functions, we know that

$$a^{log_a x} \cdot a^{log_a y} = xy$$

.

Therefore,

$$a^{log_a x + log_a y} = xy$$

.

Since $a^{log_a x + log_a y} \in \mathbb{R}^+$ (and xy too, naturally), we can apply the logarithmic function (base a) to both sides of the previous equation

$$log_a\left(a^{log_a x + log_a y}\right) = log_a(xy)$$

But, again, by the definition of the exponential and logarithmic functions, we have

$$log_a x + log_a y = log_a(xy)$$

.

Q.E.D.

Propositio

$$log_a x - log_a y = log_a \left(\frac{x}{y} \right)$$

Demonstratio

Consider the quantity $\frac{a^{log_a x}}{a^{log_a y}}$.

By a power law, we know that

$$\frac{a^{log_a x}}{a^{log_a y}} = a^{log_a x - log_a y}$$

.

But, by the definition of the exponential and logarithmic functions, we know that

$$\frac{a^{log_a x}}{a^{log_a y}} = \frac{x}{y}$$

.

Therefore,

$$a^{log_a x - log_a y} = \frac{x}{y}$$

.

Since $a^{log_a x - log_a y} \in \mathbb{R}^+$ (and $\frac{x}{y}$ too, naturally), we can apply the logarithmic function (base a) to both sides of the previous equation

$$log_a \left(a^{log_a x - log_a y} \right) = log_a \left(\frac{x}{y} \right)$$

But, again, by the definition of the exponential and logarithmic functions, we have

$$log_a x - log_a y = log_a \left(\frac{x}{y} \right)$$

.

Q.E.D.

Propositio

$$ylog_a x = log_a\left(x^y\right)$$

Demonstratio

Consider the quantity $\left(a^{log_a x}\right)^y$.

By a power law, we know that

$$\left(a^{log_a x}\right)^y = a^{ylog_a x}$$

.

But, by the definition of the exponential and logarithmic functions, we know that

$$\left(a^{log_a x}\right)^y = x^y$$

.

Therefore,

$$a^{ylog_a x} = x^y$$

.

Since $a^{ylog_a x} \in \mathbb{R}^+$ (and x^y too, naturally), we can apply the logarithmic function (base a) to both sides of the previous equation

$$log_a\left(a^{ylog_a x}\right) = log_a(x^y)$$

But, again, by the definition of the exponential and logarithmic functions, we have

$$ylog_a x = log_a\left(x^y\right)$$

.

Q.E.D.